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Quantum solvable models with $gl(2, c)$ Lie algebra symmetry embedded into the extension of unitary parasupersymmetry

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Abstract
Introducing $p - 1$ new parameters into the multilinear relations, we extend the standard unitary parasupersymmetry algebra of order p so that by embedding the quantum solvable models possessing $gl(2, c)$ Lie algebra symmetry into it, the partitions of integer numbers $p - 1$ and $\frac{1}{2}p(p - 1)$ are established. These two partitions are performed by the new parameters and the product of new parameters with their labels, respectively. The former partition is just necessary for the real form h_4; however, both of them are essential for the real forms $u(2)$ and $u(1, 1)$. By occupying these parameters with arbitrary values, the energy spectra are determined by the mean value of proposed parameters for the real form h_4 with their label weight function as well as for the real forms $u(2)$ and $u(1, 1)$ with the weight function of their squared label. So for the given energies, the multilinear behaviour of parasupercharges is not specified uniquely by varying the new parameters continuously.

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1. Introduction and motivation

In the last decades, the different quantum statistics have been widely investigated. In the three and higher dimensional spaces, parafermion and paraboson statistics have been introduced as extensions of the usual Fermi and Bose statistics [1–7]. Contrary to the usual Fermi and Bose statistics which describe one-dimensional representations of the permutation groups, parafermion and paraboson statistics describe higher dimensional representations of the same groups. Explanation of the symmetry between Fermions and bosons by supersymmetry (SUSY) [8–10] led to the natural question about the generalization of SUSY to statistics with the presence of Fermions and parabosons, parafermions and bosons as well as parafermions and parabosons.

To join the parastatistics (generalized statistics) and SUSY theories together in the framework of a unified theory in relation to the non-relativistic quantum mechanics the so-called parasupersymmetry (PSUSY), Beckers and Debergh [11, 12] proposed a superposition of the (para)bosonic Hamiltonian and the parafermionic Hamiltonian of order $p = 2$, and then their algebra for bosons and parafermions was also generalized to the arbitrary order p [13]. Rubakov and Spiridonov proposed a different PSUSY algebra for describing the symmetry between bosons and parafermions of order $p = 2$ [14]. Afterwards, Khare generalized the PSUSY of order 2 to the arbitrary order p [15–17]. In the realization of PSUSY by the shape invariant quantum solvable models such as simple harmonic oscillator, three-dimensional oscillator, Morse, Scarf I, Scarf II, generalized Pöschl–Teller and Natanzon potentials [18] as well as the quantum solvable models with symmetry of real forms of $gl(2, c)$ Lie algebra like the Landau levels problem on the flat surface with symmetry of Heisenberg Lie algebra h_4, and the motion of a charged particle on the two-dimensional sphere in the presence or absence of a magnetic monopole [19, 20], the Rubakov–Spiridonov–Khare (RSK) algebra has been much more successful than the Beckers and Debergh algebra. For this reason, the symmetry between bosons and parafermions of order p (with $p = 1, 2, \ldots$) has attracted much attention and consequently the RSK symmetry model has been the so-called standard PSUSY. In the usual supersymmetric quantum mechanics, there are two isospectral partner potentials, and symmetry generators satisfy the structural relations with bilinear products. However, in the PSUSY quantum mechanics, there are the hierarchy of $p+1$ Hamiltonian with isospectral potentials and multilinear relations including the product of $p+1$ parasupercharges. Supercharge operators are nilpotent of order 2, while parasupercharges are the nilpotent operators of order $p + 1$. Contrary to the SUSY, in general case of the arbitrary order PSUSY, the expression of the bosonic Hamiltonian in terms of parasupercharges is not directly possible.

Here, we propose an extension of RSK unitary PSUSY algebra which is realized by all quantum models containing the $gl(2, c)$ Lie algebra symmetry and not by the approach of shape invariance symmetry for the models. The symmetries of $gl(2, c)$ Lie algebra have been generally discussed for the two- and three-dimensional quantum solvable models. But the shape invariance symmetry has been used for the one-dimensional quantum solvable models. Meanwhile in the Lie algebra symmetries, the laddering generators are independent of the representation space parameters which is in contrast to the shape invariance case. For this reason, the proposed PSUSY in this paper is not realized by the models possessing shape invariance symmetry. But this does not mean that one-dimensional solvable shape-invariant models cannot realize the proposed new PSUSY. Since regarding the following two references, it appears that the quantum states of one-dimensional solvable models can also represent real forms of $gl(2, c)$. In [21, 22], a different mathematical method has been used through which the Lie algebras $su(2)$ and $su(1, 1)$ can also be represented by the quantum states of the one-dimensional solvable models such as Morse and Pöschl–Teller potentials. This method can be substituted instead of the traditional factorization method in which the hierarchy of one-dimensional partner potentials is expressed in terms of the hierarchy of superpotentials. So, in addition to the two- and three-dimensional models possessing the symmetries of $su(2)$ and $su(1, 1)$ Lie algebras, the extended PSUSY algebra proposed in this paper will be realized by the one-dimensional quantum solvable models with the same symmetries. It is also worth mentioning that some other different approaches to the PSUSY have been suggested [23, 24].

This paper has been organized as follows. In section 2 we extend the standard PSUSY algebra of arbitrary order p with new parameters such that the unitarity structure of the algebra is not violated and the well-known RSK algebra becomes a special case of the extended algebra. In section 3 we will have a short review on the three different real forms of $gl(2, c)$ Lie algebra and point out their differences in connection with the structure constants and their representation spaces. It is evident that the representation spaces can be the space of quantum states corresponding to the physical solvable models. In section 4 we introduce the parafermionic operators in terms of the raising and lowering generators of the $gl(2, c)$ complex Lie algebra. We also define the bosonic operator as a diagonal matrix whose elements are expressed in terms of the four generators of $gl(2, c)$. Then, we shall demand the realization of the algebraic relations of the extended PSUSY. It is shown that the real form h_4 causes the integer number $p-1$ to be partitioned into at most $p-1$ real parts by the proposed parameters which are not equal to each other necessarily. In addition to the mentioned partition, for the two real forms $u(2)$ and $u(1, 1)$ it is necessary the integer number $p(p-1)/2$ to be partitioned into at most $p-1$ real parts by the product of proposed parameters and their labels. So for determining the energy spectrum of the system, we will have different multilinear behaviour of parafermions with $p-2$ or $p-3$ degrees of freedom. For h_4 (and h_3) in case $p = 2$, as well as for $u(2)$ and $u(1, 1)$ in cases $p = 2$ and 3, the extended PSUSY algebra is automatically reduced to the RSK standard PSUSY. In section 5 we introduce the eigenvalue equations hierarchy of isospectrum Hamiltonians using the representation space of the quantum states of every real forms, separately. Finally, section 6 is devoted to the concluding remarks.

2. Towards the extension of standard unitary PSUSY algebra

Now, we extend the RSK standard PSUSY for the parasupercharge operators Q and $Q^\dagger$ and the bosonic operator H as

$$Q^p Q^\dagger + \beta_{p-1} Q^{p-1} Q^\dagger Q + \beta_{p-2} Q^{p-2} Q^\dagger Q^2 + \cdots + \beta_1 Q Q^\dagger Q^{p-1} + Q^\dagger Q^p = 2p Q^{p-1} H \tag{1a}$$

$$Q^{\dagger p} Q + \beta_1 Q^{\dagger p-1} Q Q^\dagger + \beta_2 Q^{\dagger p-2} Q Q^{\dagger 2} + \cdots + \beta_{p-1} Q^\dagger Q Q^{\dagger p-1} + Q Q^{\dagger p} = 2p Q^{\dagger p-1} H \tag{1b}$$

$$Q^{p+1} = Q^{\dagger p+1} = 0 \tag{1c}$$

$$[H, Q] = [H, Q^\dagger] = 0, \tag{1d}$$

where $\beta_1, \beta_2, \ldots, \beta_{p-1}$ are real constants through which the number $p-1$ is partitioned into $p-1$ real parts that are not equal to each other necessarily:

$$\Sigma_{k=1}^{p-1} \beta_k = p - 1. \tag{2}$$

The real parameters β_k can take non-integer values as well. One can verify that not only relations (1c) and (1d) are separately closure with respect to the Hermitian conjugation but also the new multilinear parts (1a) and (1b) are Hermitian conjugate of each other. Hence, the new PSUSY is unitary. Note that in the $p = 1$ case, which is the well-known SUSY, it is not necessary to introduce the parameters β_k since all of them are zero which is consistent with (2). When $\beta_1 = \beta_2 = \cdots = \beta_{p-1} = 1$, relations (1$a$), (1$b$), (1$c$) and (1$d$) are reduced to the RSK standard PSUSY algebra, and relation (2) is also satisfied. Now we focus on the realization of the extended PSUSY algebra by the quantum solvable models which have the symmetry of $gl(2, c)$ Lie algebra. One may follow the h_3 Heisenberg algebra approach of the simple harmonic oscillator to the realization of relations (1) via using the techniques presented in this paper for the h_4 Heisenberg algebra [25]. The mathematical procedure in embedding all of the real forms of the $gl(2, c)$ Lie algebra into the PSUSY algebra (1) leads to the fact that the integer number $p-1$ must be partitioned into $p-1$ real parts by $\beta_1, \beta_2, \ldots, \beta_{p-1}$ in according to (2). For these new parameters, if we define two mean values with their label weight function and the weight function of their squared label, respectively as

$$\bar{\beta} := \frac{\Sigma_{k=1}^{p-1} k\beta_k}{p(p-1)/2} \tag{3}$$

$$\bar{\bar{\beta}} := \frac{\Sigma_{k=1}^{p-1} k^2\beta_k}{p(p-1)(2p-1)/6}, \tag{4}$$

then for embedding $u(2)$ and $u(1,1)$ Lie algebras into the PSUSY algebra (1), $\bar{\beta}$ must be equal to 1. Also, the eigenvalues corresponding to the components of the bosonic Hamiltonian H are described in terms of $\bar{\beta}$ for the h_4 case and $\bar{\bar{\beta}}$ for the $u(2)$ and $u(1,1)$ cases. Thus in the cases of $u(2)$ and $u(1,1)$ in addition to partition (2) it is necessary that the integer number $\frac{p(p-1)}{2}$ is partitioned into $p-1$ real parts by the product of new parameters β_k and their labels. The products $k\beta_k$ are not equal to each other necessarily. If we demand a unit value for $\bar{\beta}$ and $\bar{\bar{\beta}}$ which is hold for the standard PSUSY then, the realization of the extended PSUSY by h_4, $u(2)$ and $u(1,1)$ requires that the number of free parameters becomes $p-3$, $p-4$ and $p-4$, respectively. Therefore, in addition to the realization of the extended PSUSY by the real forms of $gl(2, c)$ Lie algebra, one may follow the statistical description via occupation of different algebraic modes. In fact the parameters β_k can continuously vary so that for the h_4 case only constraint (2), and for the $u(2)$ and $u(1,1)$ cases both constraints (2) and $\bar{\beta} = 1$ are satisfied. So by this way, the different algebraic modes which are continuously distinguished from each other, are occupied. The occupation of different values for the free parameters β_k not only does not have any effect on the energy of the bosonic Hamiltonian H, but also proposes different algebraic multilinear relations with the same physical results. The standard PSUSY is one of these occupied modes.

3. A short review of $gl(2, c)$ Lie algebra

The most general form of the commutation relations of four generators corresponding to the $gl(2, c)$ Lie algebra in the Cartan bases is given by

$$[L_+, L_-] = aL_3 + b \tag{5a}$$

$$[L_3, L_\pm] = \pm L_\pm \tag{5b}$$

$$[1, L_+] = [1, L_-] = [1, L_3] = 0, \tag{5c}$$

where L_+ and L_- are the raising and lowering operators of the indices of the bases of the representation space, respectively. Moreover, a and b are the real structure constants. This Lie algebra has six real forms including $u(2)$, $u(1,1)$, h_4, $su(2) \oplus u(1)$, $su(1,1) \oplus u(1)$ and $iso(2) \oplus u(1)$, and they are obtained from (5) for $a > 0$ and $b \neq 0$, $a < 0$ and $b \neq 0$, $a = 0$ and $b \neq 0$, $a > 0$ and $b = 0$, $a < 0$ and $b = 0$, and $a = b = 0$, respectively. Note that the second three of them are a decomposition of the first three of them and can be obtained by choosing $b = 0$. So without missing the generality of the discussions, we shall consider only $u(2)$, $u(1,1)$ and h_4 Lie algebras.

An unitary and irreducible representation of $u(2)$ Lie algebra with $a = 2$ and $b =$ integer can be introduced by the bases $|l, m\rangle$ in which m is run as $-l - b \leqslant m \leqslant l$:

$$L_+|l, m-1\rangle = \sqrt{(l-m+1)(l+m+b)}|l, m\rangle \tag{6a}$$

$$L_-|l, m\rangle = \sqrt{(l-m+1)(l+m+b)}|l, m-1\rangle \tag{6b}$$

$$L_3|l, m\rangle = m|l, m\rangle \tag{6c}$$

$$1|l, m\rangle = |l, m\rangle. \tag{6d}$$

It must be emphasized that $l \geqslant -\frac{1}{2}(1+b)$. According to (6a) and (6b), $|l, l\rangle$ and $|l, -l-b\rangle$ are the highest and lowest bases, respectively. The representation bases are equipped with an inner product so that they are orthonormal with respect to it i.e. $\langle l, m|l, m'\rangle = \delta_{mm'}$. Thus, they constitute a Hilbert space with finite dimension $2l + 1 + b$ as $\mathcal{H}^{(l)}_{u(2)} = \mathrm{span}\{|l, m\rangle\}_{m=l}^{-l-b}$. The operators L_+ and L_- are Hermitian conjugate of each other with respect to the inner product, that is, $\langle l, m|L_+|l, m'\rangle = \langle l, m'|L_-|l, m\rangle^*$ and the operator L_3 is Hermitian: $\langle l, m|L_3|l', m\rangle = \langle l', m|L_3|l, m\rangle^*$.

For $u(1, 1)$ Lie algebra when $a = -2$ and $b =$ integer, the discrete unitary irreducible representations $\mathcal{D}^+\left(m + \frac{b}{2} + \frac{1}{2}\right)$ with $m \geqslant 0$ and $\mathcal{D}^+\left(-m + \frac{b}{2} + \frac{1}{2}\right)$ with $m \leqslant 0$ are introduced by the bases $|l, m\rangle$ with $l \geqslant |m| + \frac{b}{2} + \frac{1}{2}$ as

$$L_+|l-1, m\rangle = \sqrt{\left(l - m - \frac{b}{2} - \frac{1}{2}\right)\left(l + m - \frac{b}{2} - \frac{1}{2}\right)}|l, m\rangle \tag{7a}$$

$$L_-|l, m\rangle = \sqrt{\left(l - m - \frac{b}{2} - \frac{1}{2}\right)\left(l + m - \frac{b}{2} - \frac{1}{2}\right)}|l-1, m\rangle \tag{7b}$$

$$L_3|l, m\rangle = l|l, m\rangle \tag{7c}$$

$$1|l, m\rangle = |l, m\rangle. \tag{7d}$$

These representations have the lowest bases $\left|m + \frac{b}{2} + \frac{1}{2}, m\right\rangle$ and $\left|-m + \frac{b}{2} + \frac{1}{2}, m\right\rangle$, respectively. The infinite-dimensional Hilbert space $\mathcal{H}^{(m)}_{u(1,1)} = \mathrm{span}\{|l, m\rangle\}_{l \geqslant |m| + \frac{b}{2} + \frac{1}{2}}$ is equipped with an inner product such that the bases are orthonormal with respect to it as $\langle l, m|l', m\rangle = \delta_{ll'}$. Once again, the operators L_+ and L_- are Hermitian conjugate of each other i.e. $\langle l, m|L_+|l', m\rangle = \langle l', m|L_-|l, m\rangle^*$ and the operator L_3 is Hermitian, that is, $\langle l, m|L_3|l', m\rangle = \langle l', m|L_3|l, m\rangle^*$.

The h_4 Lie algebra with $b > 0$ and $a = 0$ has a unitary and irreducible representation through the bases $|l, m\rangle$ with the restriction $m \leqslant l$ for the non-negative integer l:

$$L_+|l, m-1\rangle = \sqrt{(l-m+1)b}|l, m\rangle \tag{8a}$$

$$L_-|l, m\rangle = \sqrt{(l-m+1)b}|l, m-1\rangle \tag{8b}$$

$$L_3|l, m\rangle = m|l, m\rangle \tag{8c}$$

$$1|l, m\rangle = |l, m\rangle, \tag{8d}$$

where $|l, l\rangle$ is the highest base. There exists an inner product so that the representation bases are orthonormal with respect to it as $\langle l, m|l, m'\rangle = \delta_{mm'}$. So, these bases constitute a finite-dimensional Hilbert space as $\mathcal{H}^{(l)}_{h_4} = \mathrm{span}\{|l, m\rangle\}_{m \leqslant l}$. As before, the operators L_+ and L_- are Hermitian conjugate of each other with respect to the inner product and the operator L_3 is Hermitian. It must be emphasized that in the Hilbert spaces $\mathcal{H}^{(l)}_{u(2)}$, $\mathcal{H}^{(m)}_{u(1,1)}$ and $\mathcal{H}^{(l)}_{h_4}$, the kets $|l, m\rangle$ may be the quantum states corresponding to the solvable models containing the symmetries of $u(2)$, $u(1, 1)$ and h_4 Lie algebras, respectively.

4. Realization of extended PSUSY by $gl(2,c)$ Lie algebra

Introducing the parafermionic operators Q and $Q^{\dagger}$ in terms of the generators L_- and L_+, and defining the bosonic operator H as a diagonal matrix in terms of $p+1$ Hamiltonians $H_1, H_2, \ldots, H_{p+1}$:

$$(Q)_{ij} = L_-\delta_{i+1\,j}, \qquad (Q^{\dagger})_{ij} = L_+\delta_{i\,j+1}, \qquad (H)_{ij} = H_i\delta_{ij}, \qquad i, j = 1, 2, \ldots, p+1, \tag{9}$$

the nilpotent equations (1c) are automatically satisfied for the operators Q and $Q^{\dagger}$. The commutation relations of the parafermionic operators with the bosonic operator i.e. (1d) for $i = 1, 2, \ldots, p$ give

$$H_i L_- = L_- H_{i+1} \tag{10a}$$

$$H_{i+1} L_+ = L_+ H_i. \tag{10b}$$

Considering definitions (9), it is seen that the parafermionic operators Q and $Q^{\dagger}$ are actually Hermitian conjugate of each other. For components of the Hermitian bosonic operator, the following explicit forms in terms of the generators of $gl(2,c)$ Lie algebra are proposed as

$$H_i = \tfrac{1}{2}(L_-L_+ + d_i L_3 + e_i) \qquad i = 1, 2, \ldots, p \tag{11a}$$

$$H_{p+1} = \tfrac{1}{2}(L_+L_- + d_{p+1} L_3 + e_{p+1}), \tag{11b}$$

where d_i and e_i are constants which can be determined by satisfying relations (10a), (10b), (1a) and (1b). Substituting the proposed relations (11a) and (11b) in equation (10a), the following recursion relations are imposed on the constants d_i and e_i:

$$d_{i+1} = d_i + a \qquad i = 1, 2, \ldots, p-1 \tag{12a}$$

$$e_{i+1} = e_i - d_i + b \qquad i = 1, 2, \ldots, p-1 \tag{12b}$$

$$d_{p+1} = d_p \tag{12c}$$

$$e_{p+1} = e_p - d_p. \tag{12d}$$

Clearly in order to obtain the above results, the commutation relations corresponding to the Lie algebra $gl(2,c)$ have been used. Note that if we use (10b) instead of (10a) then, the above results are deduced. Using (12c) and repeated applications of (12a) yields

$$d_i = d_1 + (i-1)a \qquad i = 1, 2, \ldots, p \tag{13a}$$

$$d_{p+1} = d_1 + (p-1)a. \tag{13b}$$

Applying (13a) and (12d) along with repeated application of (12b), we have

$$e_i = \frac{-a}{2}(i-1)(i-2) + (i-1)(b-d_1) + e_1 \qquad i = 1, 2, \ldots, p \tag{14a}$$

$$e_{p+1} = \frac{-a}{2}p(p-1) + (p-1)b - pd_1 + e_1. \tag{14b}$$

Hence by accepting results (13a), (13b), (14a) and (14b), equations (1d) are satisfied. The coefficients d_1 and e_1 are determined through satisfaction of relations (1a) and (1b), and we shall show that they are functions of the parameters $\beta_1, \beta_2, \ldots, \beta_{p-1}$ via $\bar{\beta}$ and $\bar{\bar{\beta}}$.

The first and second parts of the following equations are obtained from substituting (9) in (1a) and (1b), respectively:

$$L_-^p L_+ + \Sigma_{k=1}^{p-1} \beta_{p-k} L_-^{p-k} L_+ L_-^k = 2pL_-^{p-1} H_p \tag{15a}$$

$$\Sigma_{k=1}^{p-1} \beta_{p-k} L_-^{p-k} L_+ L_-^k + L_+ L_-^p = 2pL_-^{p-1} H_{p+1} \tag{15b}$$

$$\Sigma_{k=1}^{p-1} \beta_k L_+^{p-k} L_- L_+^k + L_- L_+^p = 2pL_+^{p-1} H_1 \tag{15c}$$

$$L_+^p L_- + \Sigma_{k=1}^{p-1} \beta_k L_+^{p-k} L_- L_+^k = 2pL_+^{p-1} H_2. \tag{15d}$$

One may obtain the following relation after repeated application of the commutation relations (5a) and (5b):

$$L_\pm^{p-k} L_\mp L_\pm^k = L_\pm^p L_\mp \mp akL_\pm^{p-1} L_3 - \frac{k}{2}[(k-1)a + 2b]L_\pm^{p-1} \qquad k = 0, 1, 2, \ldots, p. \tag{16}$$

Now by substituting the explicit form of H_1 and using the first relation of (16) in (15c), it appears that if we use (2) then, the coefficients of the terms possessing $L_+^p L_-$ on the both sides will automatically become equal to each other. Comparing separately the coefficients of the expressions $L_+^{p-1} L_3$ and L_+^{p-1} on the both sides, we obtain the constants d_1 and e_1 in terms of the structure constants a and b as well as two parameters $\bar{\beta}$ and $\bar{\bar{\beta}}$ as follow:

$$d_1 = -\frac{a}{2}(p-1)\bar{\beta} \tag{17a}$$

$$e_1 = \frac{(p-1)}{2}\left[a\left(\frac{\bar{\beta}}{2} - \frac{2p-1}{6}\bar{\bar{\beta}} - 1\right) - b\bar{\beta}\right]. \tag{17b}$$

Finally, by using equations (13a), (13b), (14a) and (14b), all of the constants $d_1, d_2, \ldots, d_{p+1}$ and $e_1, e_2, \ldots, e_{p+1}$ are determined as

$$d_i = -\frac{a}{2}(p-1)\bar{\beta} + a(i-1) \qquad i = 1, 2, \ldots, p \tag{18a}$$

$$d_{p+1} = d_p = -\frac{a}{2}(p-1)\bar{\beta} + a(p-1) \tag{18b}$$

$$\begin{aligned} e_i = {} & \frac{a}{2}(p-1)\left[\left(i - \frac{1}{2}\right)\bar{\beta} - \frac{2p-1}{6}\bar{\bar{\beta}} - 1\right] - \frac{a}{2}(i-1)(i-2) \\ & + b(i-1) - \frac{b}{2}(p-1)\bar{\beta} \qquad i = 1, 2, \ldots, p \end{aligned} \tag{18c}$$

$$e_{p+1} = -\frac{a}{2}(p-1)\left(p + 1 - \frac{2p+1}{2}\bar{\beta} + \frac{2p-1}{6}\bar{\bar{\beta}}\right) + b(p-1)\left(1 - \frac{\bar{\beta}}{2}\right). \tag{18d}$$

We shall show that for consistency of the above results with relations (1a) and (1b), we must choose $\bar{\beta} = 1$ when $a \neq 0$.

Although we have obtained the required results however, still relations (15a), (15b) and (15d) must be satisfied. In what follows we first check the satisfaction of (15d). Regarding the previous results, one may obtain

$$H_2 = H_1 + \frac{a}{2}L_3 + \frac{b}{2} + \frac{a}{4}(p-1)\bar{\beta}. \tag{19}$$

By using (19) on the right-hand side of (15d) and then applying (15c) in the obtained result, it is known that (15d) is automatically satisfied for $a = 0$. Also for $a \neq 0$, it is necessary to impose the following extra constraint on the real parameters $\beta_1, \beta_2, \ldots, \beta_{p-1}$:

$$\bar{\beta} = 1 \qquad \text{for} \quad a \neq 0. \tag{20}$$

Considering relations (2) and (20) for dynamical symmetry groups $U(2)$ and $U(1,1)$, it can be shown that in the cases $p = 2$ and 3 we have $\beta_1 = 1$ and $\beta_1 = \beta_2 = 1$ respectively, and consequently the extended PSUSY algebra is reduced to the RSK standard PSUSY. Thus in the cases $u(2)$ and $u(1,1)$, the extension of PSUSY is automatically accomplished for $p \geqslant 4$. In the second stage, it is investigated that how the final results i.e. relations (18) and (20) can lead to the satisfaction of (15b). Using the second equation of (16), the left-hand side of relation (15b) becomes

$$\begin{aligned}\Sigma_{k=1}^{p-1}\beta_{p-k}L_-^{p-k}L_+L_-^k + L_+L_-^p &= \left(1+\Sigma_{k=1}^{p-1}\beta_k\right)L_-^pL_+ + a\left(p+\Sigma_{k=1}^{p-1}k\beta_{p-k}\right)L_-^{p-1}L_3\\ &\quad - \left[a\Sigma_{k=1}^{p-1}\frac{k(k-1)}{2}\beta_{p-k} + a\frac{p(p-1)}{2} - b\Sigma_{k=1}^{p-1}k\beta_{p-k} - bp\right]L_-^{p-1}.\end{aligned} \tag{21}$$

We substitute the explicit form of H_{p+1} on the right-hand side of (15b), and then the generators are put in order like (26) using the commutation relations of $gl(2,c)$ Lie algebra. Now, the obtained result is compared with (26), then it appears that with accepting only (2), the known results (18b) and (18d) are obtained for d_{p+1} and e_{p+1}, respectively. So, relation (15b) is also satisfied. To complete our considerations of the realization of the extended PSUSY (1), in the third and final stage, we must make sure that (15a) is also satisfied. It is easily seen that the commutation relations of $gl(2,c)$ Lie algebra give

$$H_p = H_{p+1} - \frac{a}{2}L_3 + \frac{a}{2}(p-1)\left(1-\frac{\bar{\beta}}{2}\right) - \frac{b}{2}. \tag{22}$$

Using (22) on the right-hand side of (15a) and imposing (15b) in it as well, then from equation (18) we find that relation (15a) is automatically satisfied for $a = 0$. Furthermore, when $a \neq 0$, one have to accept constraint (20) again for the parameters $\beta_1, \beta_2, \ldots, \beta_{p-1}$.

The satisfaction of four relations (15a), (15b), (15c) and (15d) causes a question in connection with the unitarity of our realization from PSUSY. Indeed, it seems that only realization of relations (15a) and (15d) for $a \neq 0$ requires to accept constraint (20). On the other hand, one may check that the left-hand sides of relations (15a) and (15c) are Hermitian conjugate of each other as well as the left-hand sides of (15b) and (15d). Moreover, from repeated application of the commutation relations of $gl(2,c)$ Lie algebra together with using (18a), (18b), (18c) and (18d), we get

$$H_pL_+^{p-1} = L_+^{p-1}H_1 \tag{23a}$$

$$H_{p+1}L_+^{p-1} = L_+^{p-1}H_2. \tag{23b}$$

The above relations show that the right-hand sides of (15a) and (15c) as well as of (15b) and (15d) are Hermitian conjugate of each other. So, equations (15a) and (15c) are Hermitian conjugate of each other. The same fact is hold for (15b) and (15d). Therefore, the unitary structure of the theory requires that constraint (20) when $a \neq 0$ is hold for the satisfaction of not only (15a) and (15d) but also (15b) and (15c). In fact, one can put in order the generators of $gl(2,c)$ Lie algebra on the both sides of equations (15b) and (15c) such that constraint (20) is concluded. Using (18a), constraint (20) when $a \neq 0$ leads to

$$\Sigma_{i=1}^{p}d_i = 0. \tag{24}$$

Note that for $a = 0$, relation (24) is automatically hold without satisfaction of constraint (20). This means that for a dynamical symmetry group H_4 (as well as H_3) in case $p = 3$, one of the parameters β_1 and β_2 is free, and consequently the extension of PSUSY is accomplished for $p \geqslant 3$. So, relation (24) could be considered as a necessary condition for the unitarity of the new PSUSY. There is a similar condition for the standard PSUSY as well [17, 19]. Also, we have

$$\Sigma_{i=1}^{p} e_i = \frac{-a}{12} p(p-1)(2-p+(2p-1)\bar{\bar{\beta}}) \qquad \text{for} \quad a \neq 0 \tag{25a}$$

$$\Sigma_{i=1}^{p} e_i = \frac{b}{2} p(p-1)(1-\bar{\beta}) \qquad \text{for} \quad a = 0. \tag{25b}$$

Result (25a) is independent of the structure constant b such as the situation in the standard PSUSY. Meanwhile, in the standard case, constraint (20) is automatically hold and, consequently expression (25b) vanishes which is the well-known unitary condition of the standard PSUSY algebra for the shape invariant models. For $a = 0$, the reason for doing this comparison is the fact that one may use the formulation of this paper for the simple harmonic oscillator algebra h_3 which is also a one-dimensional shape invariant model. Using results (18), the familiar relations of the standard PSUSY between solvable Hamiltonians $H_1, H_2, H_3, \ldots, H_{p+1}$ as independent of the parameters $\beta_1, \beta_2, \ldots, \beta_{p-1}$ may again be found:

$$H_{i+1} = H_i + \frac{b}{2} \qquad \text{for} \quad h_4 (\text{and } h_3) \tag{26a}$$

$$H_{i+1} = H_i + \frac{a}{2} L_3 + \frac{a}{4}(p-2i+1) + \frac{b}{2} \qquad \text{for} \quad u(2) \quad \text{and} \quad u(1,1). \tag{26b}$$

This verifies that the solvable Hamiltonians with dynamical symmetry groups H_4 (and H_3), $U(2)$ and $U(1,1)$ realize the extended PSUSY algebra. Therefore, the dynamical symmetry groups with their corresponding Lie algebras as h_4 (and h_3) as well as $u(2)$ and $u(1,1)$ are embedded into non-unique PSUSY algebras with arbitrary orders $p \geqslant 3$ and $p \geqslant 4$, respectively.

5. Isospectrum Hamiltonians in the representation spaces of $u(2)$, $u(1,1)$ and h_4

Now we can construct isospectrum Hamiltonians for each of the quantum states spaces $\mathcal{H}_{u(2)}^{(l)}$, $\mathcal{H}_{u(1,1)}^{(m)}$ and $\mathcal{H}_{h_4}^{(l)}$ which are the representation spaces of their corresponding Lie algebras with the commutation relations (5a), (5b) and (5c) for the cases $a = 2$, $a = -2$ and $a = 0$, respectively. Using the representations of these algebras by their Hilbert spaces given by relations (6), (7) and (8), we obtain the following eigenvalue equations for $p+1$ Hamiltonians $H_1, H_2, \ldots, H_{p+1}$ given in (11a) and (11b):

$$H_m |l, m-1\rangle = E_{u(2)}(l, p, \bar{\bar{\beta}}) |l, m-1\rangle \qquad m = 1, 2, \ldots, p+1 \tag{27a}$$

$$H_l |l-1, m\rangle = E_{u(1,1)}(m, p, \bar{\bar{\beta}}) |l-1, m\rangle \qquad l = 1, 2, \ldots, p+1 \tag{27b}$$

$$H_m |l, m-1\rangle = E_{h_4}(l, p, \bar{\beta}) |l, m-1\rangle \qquad m = 1, 2, \ldots, p+1, \tag{27c}$$

in which the spectra are independent of m, l and m, respectively:

$$E_{u(2)}(l, p, \bar{\bar{\beta}}) = \frac{1}{2}\left[l^2 - \frac{1}{2}(1+b)(p-2l-1) - (p-1)(2p-1)\frac{\bar{\bar{\beta}}}{6}\right] \tag{28a}$$

$$E_{u(1,1)}(m, p, \bar{\bar{\beta}}) = \frac{1}{2}\left[\frac{b^2-1}{4} - m^2 + \frac{p}{2}(1-b) + (p-1)(2p-1)\frac{\bar{\bar{\beta}}}{6}\right] \tag{28b}$$

$$E_{h_4}(l, p, \bar{\beta}) = b\left[l - (p-1)\frac{\bar{\beta}}{2}\right]. \tag{28c}$$

The above relations, that in their proofs we have used (18*a*), (18*b*), (18*c*) and (18*d*), have differences as well as similarities with each others. The dimension of the representation space $\mathcal{H}^{(l)}_{u(2)}$ is finite since the $u(2)$ Lie algebra is compact. So, the order of the PSUSY algebra, say, p must obey the restriction $p \leqslant l$ with the assumption $b > 0$. In other words in the Hilbert representation space $\mathcal{H}^{(l)}_{u(2)}$, one can construct at most $l+1$ isospectrum Hamiltonians. Although the Hilbert representation spaces $\mathcal{H}^{(m)}_{u(1,1)}$ and $\mathcal{H}^{(l)}_{h_4}$ are infinite-dimensional spaces however, for the former with the assumption $b \leqslant -1-2|m|$, the number of isospectrum Hamlitonians which is $p+1$ dose not have any restriction. But for the latter, at most $l+1$ isospectrum Hamiltonians can be constructed once again. For the representation spaces $\mathcal{H}^{(l)}_{u(2)}$ and $\mathcal{H}^{(m)}_{u(1,1)}$ that $a \neq 0$ in their commutation relations, both of their spectra are functions of $\bar{\bar{\beta}}$ (in both cases $\bar{\beta}$ must be equal to one). However, the spectrum corresponding to $\mathcal{H}^{(l)}_{h_4}$ for which $a = 0$, is a function of $\bar{\beta}$.

It is evident that for the spectra of $u(2)$ and $u(1,1)$, $\bar{\bar{\beta}}$ is a function of $p-3$ free parameters; however, for the spectrum of h_4, $\bar{\beta}$ is a function of $p-2$ free parameters:

$$\bar{\bar{\beta}} = \frac{3(3p-4)}{p(2p-1)} + \frac{\Sigma_{k=3}^{p-1}(k-1)(k-2)\beta_k}{p(p-1)(2p-1)/6} \tag{29a}$$

$$\bar{\beta} = \frac{2}{p} + \frac{\Sigma_{k=2}^{p-1}(k-1)\beta_k}{p(p-1)/2}. \tag{29b}$$

In standard PSUSY, we have $\beta_1 = \beta_2 = \cdots = \beta_{p-1} = 1$, so the quantities $\bar{\bar{\beta}}$ and $\bar{\beta}$ are equal to unit. Hence, the physical results (27) and (28) are actually reduced to the physical results of the standard PSUSY. Clearly, if we demand the given values for $\bar{\bar{\beta}}$ and $\bar{\beta}$ then, the number of free parameters will be reduced to $p-4$ and $p-3$, respectively. Thus, apart from obtaining the physical results similar to the standard PSUSY case, we can use the extra free parameters in order to develop the embedding of the $gl(2,c)$ Lie algebra into the PSUSY algebra which will be discussed in the following section.

Theoretically, the magnetic monopole problem has attracted very much attention, since its wavefunctions can be calculated exactly via the dynamical symmetry group $SU(2)$. Here, as a physical application of the extension, we briefly consider how the extended PSUSY can be realized by $su(2)$-generators of the magnetic monopole model. The angular momentum operators (as example see [26, 27])

$$L^N_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm\frac{\partial}{\partial\theta} + \mathrm{i}\cot\theta\frac{\partial}{\partial\phi} - q\frac{1-\cos\theta}{\sin\theta}\right) \qquad L^N_3 = -\mathrm{i}\frac{\partial}{\partial\phi} - q \tag{30a}$$

$$L^S_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm\frac{\partial}{\partial\theta} + \mathrm{i}\cot\theta\frac{\partial}{\partial\phi} - q\frac{1+\cos\theta}{\sin\theta}\right) \qquad L^S_3 = -\mathrm{i}\frac{\partial}{\partial\phi} + q, \tag{30b}$$

which satisfy the following commutation relations:

$$\left[L^{N(S)}_+, L^{N(S)}_-\right] = 2L^{N(S)}_3, \qquad \left[L^{N(S)}_3, L^{N(S)}_\pm\right] = \pm L^{N(S)}_\pm, \tag{31}$$

represent $su(2)$ Lie algebra via monopole harmonics $Y^{N(S),q}_{lm}(\theta,\phi)$ as

$$L^{N(S)}_+ Y^{N(S),q}_{lm}(\theta,\phi) = \sqrt{(l+m+1)(l-m)}\,Y^{N(S),q}_{lm+1}(\theta,\phi) \tag{32a}$$

$$L_-^{N(S)} Y_{lm}^{N(S),q}(\theta, \phi) = \sqrt{(l-m+1)(l+m)}\, Y_{lm-1}^{N(S),q}(\theta, \phi) \tag{32b}$$

$$L_3^{N(S)} Y_{lm}^{N(S),q}(\theta, \phi) = m Y_{lm}^{N(S),q}(\theta, \phi). \tag{32c}$$

The indices N and S label the local coordinates in open neighbourhoods of north and south poles as $0 \leqslant \theta < \pi$ and $0 < \theta \leqslant \pi$, respectively. ϕ is auxiliary variable: $0 \leqslant \phi < 2\pi$. q is the magnetic charge and it has been located at the origin of the spherical coordinate system. In atomic units ($m = e = \hbar = 1$), Hamiltonian corresponding to the motion of a electron in the presence of the magnetic charge q is irreducibly represented by monopole harmonics with a given orbital momentum l:

$$H^{N(S)} Y_{lm}^{N(S),q}(\theta, \phi) = l(l+1) Y_{lm}^{N(S),q}(\theta, \phi), \tag{33}$$

in which the explicit forms of Hamiltonians are as follows:

$$H^{N(S)} = \frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \pm \frac{2iq}{1 \pm \cos\theta}\frac{\partial}{\partial\phi} + \frac{2q^2}{1 \pm \cos\theta}. \tag{34}$$

All monopole harmonics corresponding to the quantum number l make an irreducible representation of $su(2)$ Lie algebra via relations (32). According to above discussions, $p+1$ bosonic Hamiltonians

$$\begin{aligned} H_m^{N(S)} = \frac{1}{2}\Bigg[& L_-^{N(S)} L_+^{N(S)} + (2m - p - 1) L_3^{N(S)} \\ & + (p-1)\left(m - \frac{3}{2} - (2p-1)\frac{\bar{\bar{\beta}}}{6}\right) - (m-1)(m-2)\Bigg] \qquad 1 \leqslant m \leqslant p+1, \end{aligned} \tag{35}$$

satisfy the following eigenvalue equations on the magnetic monopole harmonics of the orbital momentum l with limitation $p \leqslant l$ for PSUSY order:

$$H_m^{N(S)} Y_{lm-1}^{N(S),q}(\theta, \phi) = \frac{1}{2}\left[l^2 - \frac{1}{2}(p - 2l - 1) - (p-1)(2p-1)\frac{\bar{\bar{\beta}}}{6}\right] Y_{lm-1}^{N(S),q}(\theta, \phi). \tag{36}$$

Therefore, for a given energy, $p+1$ isospectrum Hamiltonians can be embedded in the extended PSUSY algebra with $p-4$ free parameters of β_k's.

6. Concluding remarks

There are many quantum-mechanical models whose solutions represent one of the real forms of $gl(2, c)$ Lie algebra symmetry. For example, the motion of a free or charged particle on a sphere (or hyperbolic) surface in the absence or presence of a magnetic field of a monopole, the motion of a charged particle on a two-dimensional flat plane in the presence of a uniform magnetic field (Landau problem) or even one-dimensional models with the Morse and Pöschl–Teller potentials. Existence of such symmetry leads to the fact that we can obtain the ground state by solving a first-order differential equation; then, we may calculate the other states or the representation space bases by using an algebraic method. In fact our knowledge about the symmetry gives rise to a simplification in solving the quantum-mechanical problems so that it is not necessary to integrate twice from the equation of motion. The description of symmetry between bosons and parafermions by the representation space corresponding to one of the real forms of the $gl(2, c)$ Lie algebra can be used for many quantum-mechanical models.

Embedding $gl(2, c)$ Lie algebra into the extended PSUSY algebra requires that in both cases $a = 0$ and $a \neq 0$, the positive integer $p-1$ must be partitioned into $p-1$ real parts β_k

which are not equal to each other necessarily. So in practice only for $a = 0$, not only $p \geqslant 2$ but also the number of free parameters must be equal to $p - 2$. In the case $a \neq 0$, the integer number $\frac{1}{2}p(p-1)$ must also be partitioned into $p - 1$ real parts not necessarily equal to each other through the product of the parameters β_k with their labels, i.e. $k\beta_k$. In fact constraint (20) is an additional partition which must again be satisfied by the parameters $\beta_1, \ldots, \beta_{p-1}$. Therefore, the number of free parameters is reduced to $p - 3$ for the case $a \neq 0$. For the cases $a = 0$ and $a \neq 0$, the energy spectra are functions from the mean value of the new parameters with their label weight function and the weight function of squared label, respectively, say, $\bar{\beta}$ and $\bar{\bar{\beta}}$. These functions generate $p - 2$ and $p - 3$ degrees of freedom in describing the energy spectrum of the quantum systems possessing the symmetries of h_4 and $u(2)$ ($u(1, 1)$) Lie algebras, respectively. If a given value, for example unit, is demanded for $\bar{\beta}$ and $\bar{\bar{\beta}}$ then, the number of degrees of freedom will be $p - 3$ and $p - 4$, respectively.

In standard PSUSY, we have $\beta_1 = \beta_2 = \cdots = \beta_{p-1} = 1$, so the mean functions are equal to 1. Therefore, the algebraic relations of the standard PSUSY can be considered as an occupied state from different algebraic modes. For instance, if we would like that the occupation of the states with the same energy leads to the minimization of the number of the terms in the multilinear parts (1*a*) and (1*b*), then the following relations (along with their Hermitian conjugates) play the same role in realization of the algebra by the quantum solvable models with the symmetry of $gl(2, c)$ Lie algebras:

$$Q^pQ^\dagger + \frac{(p-1)(p-2)(p-3)}{6}Q^3Q^\dagger Q^{p-3} + \frac{p-1}{6}(-2p^2+13p-18)Q^2Q^\dagger Q^{p-2} + \frac{p-1}{6}(p^2-8p+18)QQ^\dagger Q^{p-1} + Q^\dagger Q^p = 2pQ^{p-1}H \tag{37}$$

$$Q^pQ^\dagger + \frac{p-1}{3}Q^{p-1}Q^\dagger Q + \frac{(p-1)(p-2)}{6}Q^2Q^\dagger Q^{p-2} + \frac{(p-1)(6-p)}{6}QQ^\dagger Q^{p-1} + Q^\dagger Q^p = 2pQ^{p-1}H. \tag{38}$$

Equations (37) and (38) along with their Hermitian conjugates present two different multilinear behaviour of the parafermionic operators Q and $Q^\dagger$ with respect to each other which can be considered and occupied as different statistical modes. Nevertheless, their bosonic Hamiltonians have the same energies. Moreover, it must be emphasized that the above relations describe two different multilinear behaviour for the Heisenberg Lie algebra h_4 since in addition to equation (20), equation (2) is also satisfied. It is evident that the different modes for the multilinear relations of the PSUSY algebra realized by the symmetries of $u(2)$ and $u(1, 1)$ Lie algebras, include also h_4 necessarily; however, its inverse is not true.

Actually, multilinear behaviour of the parafermions Q and $Q^\dagger$ can be considered as a superposition of all the modes consisting of choosing different values for the continuous parameters $\beta_1, \ldots, \beta_{p-1}$. Our investigations show that for the usual SUSY, one cannot describe the bilinear relations by the parameters such as β_k with the values other than 1. For this reason in the SUSY associated with the usual Fermi and Bose statistics, there is a unique bilinear relation between the supercharges Q and $Q^\dagger$ while for the symmetry between bosons and the parafermions of order $p \geqslant 3$, there is not such a restriction on the multilinear relation of parasupercharges Q and $Q^\dagger$.

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